1A.1 Estimation of dense-gas viscosity

Table D.1 gives $T_c = 126.2\ \text{K}$, $p_c = 33.5\ \text{atm}$, and $\mu_c = 180 \times 10^{-6}\ \text{g/cm} \cdot \text{s}$ for N_2. The reduced conditions for the viscosity estimate are then

$$p_r = \frac{p}{p_c} = \frac{\left(1000 + 14.7\ \text{lb}_f/\text{in.}^2\right)}{\left(33.5\ \text{atm}\right)\left(14.7\ \text{lb}_f/\text{in.}^2 \cdot \text{atm}\right)} = 2.06 \tag{A}$$

$$T_r = \frac{T}{T_c} = \frac{273\ \text{K} + \left(68 - 32°\text{F}\right)\left(1.8\ \text{K}/°\text{F}\right)}{126.2\ \text{K}} = 2.32 \tag{B}$$

At this reduced state, Fig. 1.5-1 gives $\mu_r = 1.15$. Hence the predicted viscosity is $\mu = \mu_r \mu_c = \left(180 \times 10^{-6}\ \text{g/cm} \cdot \text{s}\right)\left(1.15\right)$ $= 2.07 \times 10^{-4}\ \text{g/cm} \cdot \text{s}$. This result is then converted into the requested units by use of Table E.3-4:

$$\mu = \left(2.07 \times 10^{-4}\ \text{g/cm} \cdot \text{s}\right)\left(\frac{6.7197 \times 10^{-2}\ \text{lb}_m/\text{ft} \cdot \text{s}}{\text{g/cm} \cdot \text{s}}\right) = 1.4 \times 10^{-5}\ \text{lb}_m/\text{ft} \cdot \text{s} \tag{C}$$

1A.2 Estimation of the viscosity of methyl fluoride

CH_3F has molecular weight $M = 16.04 - 1.008 + 19.00$ $= 34.03 \text{ g/g-mol}$, $T_c = 4.55 + 273.15 = 277.70 \text{ K}$, $p_c = 58.0 \text{ atm}$, and $\tilde{V}_c = (34.03 \text{ g/g-mol})/(0.300 \text{ g/cm}^3) = 113.4 \text{ cm}^3/\text{g-mol}$. The critical viscosity is then estimated as

$$\mu_c = 61.6(34.03 \cdot 277.70)^{1/2}(113.4)^{-2/3} = 255.6 \text{ micropoise} \qquad \text{(A)}$$

using Eq. 1.5-1(a), and

$$\mu_c = 7.70(34.03)^{1/2}(58.0)^{2/3}(277.70)^{-1/6} = 263.5 \text{ micropoise} \qquad \text{(B)}$$

using Eq. 1.5-1(b). The reduced conditions for the viscosity estimate are $T_r = (370 + 273.15 \text{ K})/(277.70 \text{ K}) = 2.32$, $p_r = (120 \text{ atm})/(58.0 \text{ atm})$ $= 2.07$, and the predicted valued of μ_r from Fig. 1.5-1 is 1.1. The resulting predicted value of the viscosity is

$$\mu = \mu_r \mu_c = (1.1)(255.6 \times 10^{-6} \text{ poise}) = 2.8 \times 10^{-4} \text{ g/cm} \cdot \text{s} \qquad \text{(C)}$$

using Eq. 1.5-1(a), and

$$\mu = \mu_r \mu_c = (1.1)(263.5 \times 10^{-6} \text{ poise}) = 2.9 \times 10^{-4} \text{ g/cm} \cdot \text{s} \qquad \text{(D)}$$

using Eq. 1.5-1(b).

1A.3 Computation of the viscosities of gases at low density

Equation 1.6-14, with molecular parameters from Table D.1 and collision integrals from Table D.2, gives the following results:

For O_2: $M = 32.00\ \text{g/g-mol}$, $\sigma = 3.433\ \text{Å}$, and $\varepsilon/\kappa = 113\ \text{K}$. Then at 20°C, $\kappa T/\varepsilon = (293.15\ \text{K})/(113\ \text{K}) = 2.594$ and $\Omega_\mu = 1.086$. Equation 1.6-14 then gives

$$\mu = 2.6693\times 10^{-5}\frac{\sqrt{(32.00)(293.15)}}{(3.433)^2(1.086)}$$
$$= 2.02\times 10^{-4}\ \text{g/cm}\cdot\text{s}\frac{(10^{-1}\ \text{Pa}\cdot\text{s})}{(\text{g/cm}\cdot\text{s})}\frac{(10^{3}\ \text{mPa}\cdot\text{s})}{(\text{Pa}\cdot\text{s})}$$
$$= 2.02\times 10^{-2}\ \text{mPa}\cdot\text{s} \qquad \text{(A)}$$

The reported value in Table 1.4-2 is $2.04\times 10^{-2}\ \text{mPa}\cdot\text{s}$.

For N_2: $M = 28.01\ \text{g/g-mol}$, $\sigma = 3.667\ \text{Å}$, and $\varepsilon/\kappa = 99.8\ \text{K}$. Then at 20°C, $\kappa T/\varepsilon = (293.15\ \text{K})/(99.8\ \text{K}) = 2.594$ and $\Omega_\mu = 1.0447$. Equation 1.6-14 then gives

$$\mu = 2.6693\times 10^{-5}\frac{\sqrt{(28.01)(293.15)}}{(3.667)^2(1.0447)}$$
$$= 1.72\times 10^{-4}\ \text{g/cm}\cdot\text{s}\frac{(10^{-1}\ \text{Pa}\cdot\text{s})}{(\text{g/cm}\cdot\text{s})}\frac{(10^{3}\ \text{mPa}\cdot\text{s})}{(\text{Pa}\cdot\text{s})}$$
$$= 1.72\times 10^{-2}\ \text{mPa}\cdot\text{s} \qquad \text{(B)}$$

The reported value in Table 1.4-2 is $1.75\times 10^{-2}\ \text{mPa}\cdot\text{s}$.

For CH_4: $M = 16.04\ \text{g/g-mol}$, $\sigma = 3.780\ \text{Å}$, and $\varepsilon/\kappa = 154\ \text{K}$. Then at 20°C, $\kappa T/\varepsilon = (293.15\ \text{K})/(154\ \text{K}) = 1.904$ and $\Omega_\mu = 1.197$. Equation 1.6-14 then gives

$$\mu = 2.6693\times 10^{-5}\frac{\sqrt{(16.04)(293.15)}}{(3.780)^2(1.197)}$$

$$= 1.07 \times 10^{-4} \text{ g/cm} \cdot \text{s} \frac{\left(10^{-1} \text{ Pa} \cdot \text{s}\right)}{\left(\text{g/cm} \cdot \text{s}\right)} \frac{\left(10^{3} \text{ mPa} \cdot \text{s}\right)}{\left(\text{Pa} \cdot \text{s}\right)}$$

$$= 1.07 \times 10^{-2} \text{ mPa} \cdot \text{s} \qquad \text{(B)}$$

The reported value in Table 1.4-2 is 1.09×10^{-2} mPa · s.

1A.4 Estimation of liquid viscosity

For Eq. 1.7-1, we need the following quantities at $0°C$ and $100°C$:

T (K)	273.15	373.15
$\tilde{N}h/\tilde{V}$ $(g/cm \cdot s)$	2.22×10^{-4}	2.12×10^{-4}
$\exp(3.8T_b/T)$	179.7	44.70

The predicted liquid viscosity is then

μ $(g/cm \cdot s)$ 0.0398 0.0095

The corresponding experimental values from Table 1.4-1 are

μ $(g/cm \cdot s)$ 0.01787 0.002821

The values predicted by Eq. 1.7-1 are in poor agreement with the experimental values. This is not surprising, since the empirical formula in Eq. 1.7-1 is inaccurate for water and other associated liquids.

1A.5 Molecular velocity and mean free path

From Eq. 1.7-1, the mean molecular velocity in O_2 is

$$\bar{u} = \sqrt{\frac{8RT}{\pi M}} = \sqrt{\frac{8\left(8.31451\times 10^{7}\ \text{g}\cdot\text{cm}^2/\text{s}^2\cdot\text{g-mol}\cdot\text{K}\right)(273.2\ \text{K})}{\pi\left(32.00\ \text{g/g-mol}\right)}}$$

$$= 4.25\times 10^{4}\ \text{cm/s} \qquad \text{(A)}$$

From Eq. 1.7-3, the mean free path in O_2 at 1 atm and $273.2\ \text{K}$ is

$$\lambda = \frac{RT}{\sqrt{2}\pi d^2 p\tilde{N}} = \frac{\left(82.0578\ \text{cm}^3\text{atm/g-mol}\cdot\text{K}\right)(273.2\ \text{K})}{\sqrt{2}\pi\left(3\times 10^{-8}\ \text{cm}\right)^2(1\ \text{atm})\left(6.02214\times 10^{23}\ \text{g-mol}^{-1}\right)}$$

$$= 9.3\times 10^{-6}\ \text{cm} \qquad \text{(B)}$$

Hence the ratio of the mean free path to the molecular diameter is

$$\frac{9.3\times 10^{-4}\ \text{cm}}{3\times 10^{-8}\ \text{cm}} = 3.1\times 10^{4} \qquad \text{(C)}$$

at these conditions. In the liquid state, on the other hand, the corresponding ratio would be of the order of unity or even less.

1B.1 Velocity profiles and stress components

(a) $\tau_{xy} = \tau_{yx} = -\mu b$, and all other τ_{ij} are zero.

$\rho v_x v_x = \rho b^2 y^2$, and all other $\rho v_i v_j$ are zero.

(b) $\tau_{xy} = \tau_{yx} = -2\mu b$, and all other τ_{ij} are zero.

$\rho v_x v_x = \rho b^2 y^2$, $\rho v_x v_y = \rho v_y v_x = \rho b^2 xy$, $\rho v_y v_y = \rho b^2 x^2$, and all other $\rho v_i v_j$ are zero.

(c) All τ_{ij} are zero.

$\rho v_x v_x = \rho b^2 y^2$, $\rho v_x v_y = \rho v_y v_x = -\rho b^2 xy$, $\rho v_y v_y = \rho b^2 x^2$ and all other $\rho v_i v_j$ are zero.

(d) $\tau_{xx} = \tau_{yy} = \mu b$, $\tau_{zz} = -2\mu b$, and all others are zero. The components of $\rho \mathbf{vv}$ may be given in the matrix:

$$\rho \mathbf{vv} = \begin{pmatrix} \rho v_x v_x = \frac{1}{4}\rho b^2 x^2 & \rho v_x v_y = \frac{1}{4}\rho b^2 xy & \rho v_x v_z = -\frac{1}{2}\rho b^2 xz \\ \rho v_y v_x = \frac{1}{4}\rho b^2 xy & \rho v_y v_y = \frac{1}{4}\rho b^2 y^2 & \rho v_y v_z = -\frac{1}{2}\rho b^2 yz \\ \rho v_z v_x = -\frac{1}{2}\rho b^2 xz & \rho v_z v_y = -\frac{1}{2}\rho b^2 yz & \rho v_z v_z = \rho b^2 z^2 \end{pmatrix}$$

1B.2 A fluid in a state of rigid rotation

(a) A particle within a rigid body rotating with an angular velocity vector **w** has a velocity given by $\mathbf{v} = [\mathbf{w} \times \mathbf{r}]$. If the angular velocity vector is in the +*z* direction, then there are two nonzero velocity components given by $v_x = -w_z y$ and $v_y = +w_z x$. Hence the magnitude of the angular velocity vector is *b* in Problem 1B.1(*c*).

(b) For the velocity components of Problem 1B.1(*c*),

$$\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} = 0 \quad \text{and} \quad \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = 2b \tag{A}$$

(c) In Eqs. 1.2-7 through 12, we employ only the linear *symmetric* combinations of derivatives of the velocity, so that in pure rotation there would be no viscous forces present. In (*b*) we see that the antisymmetric combination is nonzero in a purely rotational motion.

2A.1 Thickness of a falling film

(a) The volume flow rate w/ρ per unit wall width W is obtained from Eq. 2.2-26 thus

$$\frac{w}{\rho W} = \frac{\nu \mathrm{Re}}{4} = \frac{\left(1.0037\times 10^{-2}\ \mathrm{cm/s^2}\right)(10)}{4} = 2.509\times 10^{-2}\ \mathrm{cm/s^2} \qquad \text{(A)}$$

Here the kinematic viscosity ν for liquid water at $20\,^\circ\mathrm{C}$ was obtained fom Table 1.4-1. Since 1 ft = 2.54 cm, 1 hr = 3600 s, and 1 gal = (231.00 $\mathrm{in}^3)\times\left(2.54\ \mathrm{cm/in}\right)^3 = 3785.4\ \mathrm{cm}^3$ (see Appendix E), the result in the requested units is

$$\frac{w}{\rho W} = \left(0.02509\frac{\mathrm{cm}^2}{\mathrm{s}}\right)\left(\frac{1}{3785.3}\frac{\mathrm{gal}}{\mathrm{cm}^3}\right)\left(30.48\ \frac{\mathrm{cm}}{\mathrm{ft}}\right)\left(3600\frac{\mathrm{s}}{\mathrm{hr}}\right)$$

$$= 0.727\frac{\mathrm{gal}}{\mathrm{hr}\cdot\mathrm{ft}} \qquad \text{(B)}$$

(b) The film thickness is calculated from Eq. 2.2-27 and Eq. A as

$$\delta = \left(\frac{3\nu}{g\cos\beta}\frac{w}{\rho W}\right)^{1/3} = \left(\frac{3\nu}{g\cos\beta}\frac{\nu \mathrm{Re}}{4}\right)^{1/3}$$

$$= \left[\frac{3\left(0.010037\ \mathrm{cm^2/s}\right)}{\left(980.665\ \mathrm{cm/s^2}\right)(1.0)}\left(2.509\ \mathrm{cm^2/s}\right)\right]^{1/3}$$

$$= 0.009167\ \mathrm{cm} = 0.00361\ \mathrm{in} \qquad \text{(C)}$$

2A.2 Determination of capillary radius by flow measurement

Assuming the flow to be laminar, we solve Eq. 2.3-23 for the capillary radius to get

$$R = \left(\frac{8\mu L w}{\pi \rho (\mathscr{P}_0 - \mathscr{P}_L)} \right)^{1/4} = \left(\frac{8\nu L w}{\pi (\mathscr{P}_0 - \mathscr{P}_L)} \right)^{1/4}$$

$$= \left[\frac{8\left(4.03 \times 10^{-5}\ \text{m}^2/\text{s}\right)(0.5002\ \text{m})\left(2.997 \times 10^{-3}\ \text{kg/s}\right)}{(3.1416)\left(4.829 \times 10^{5}\ \text{kg/m} \cdot \text{s}^2\right)} \right]^{1/4}$$

$$= \left(3.186 \times 10^{-13}\right)^{1/4} = 7.51 \times 10^{-4}\ \text{m} = 7.51 \times 10^{-2}\ \text{cm} \qquad \text{(A)}$$

We must next get the Reynolds number in order to establish the fact that the flow is laminar.

$$\text{Re} = \frac{D\langle v_z \rangle \rho}{\mu} = \frac{4w}{\pi D \mu} = \frac{2w}{\pi R \nu \rho}$$

$$= \frac{2}{\pi} \frac{\left(2.997 \times 10^{-3}\ \text{kg/s}\right)}{\left(7.51 \times 10^{-4}\ \text{m}\right)\left(4.03 \times 10^{-5}\ \text{m}^2/\text{s}\right)\left(0.9552 \times 10^{3}\ \text{kg/m}^3\right)}$$

$$= 66.0 \qquad \text{(B)}$$

Thus our assumption of laminar flow has been validated. Since the entrance length $L_e = 0.035 D\text{Re} = 0.35\ \text{cm}$ is less than L, the entrance-effect correction to R is at most of the order of $\left| \left[1 - (L_e/L)\right]^{1/4} - 1 \right|$, or 0.2 percent of R in the present example.

Difficulties in this method of determination of the capillary radius include: (1) inability to account for departures from a straight, circular cylindrical wall geometry, and (2) inability to account for inadvertent spatial and temporal variations of temperature, hence of the fluid density and viscosity.

A simpler method of measuring the capillary radius is to measure the length and mass m of a small slug of liquid mercury (or another liquid of known density) injected into the tube, and to calculate the mean radius R of the slug from $\left(m/\rho\pi L\right)^{1/2}$, on the assumption that the slug is a right circular cylinder. This method allows for comparisons of the mean R values for various intervals of the tube length.

2A.3 Volume rate of flow through an annulus

Assuming the flow to be laminar, we use Eq. 2.4-18 to calculate the volume flow rate $Q = w/\rho$ for the following data

$$\kappa = 0.495/1.1 = 0.45$$
$$R = 1.1 \text{ in} = (1.1 \text{ in}/12 \text{ in/ft}) = (1.1/12) \text{ ft}$$
$$\mu = (136.8 \text{ lb}_m/\text{ft}\cdot\text{hr})(1 \text{ hr}/3600 \text{ s}) = 3.80\times10^{-2} \text{ lb}_m/\text{ft}\cdot\text{s}$$
$$L = 27 \text{ ft}$$
$$\mathscr{P}_0 - \mathscr{P}_L = (5.39 \text{ psi})\left(4.330\times10^3 \left(\text{poundals}/\text{ft}^2\right)/\text{psi}\right)$$
$$= 2.497\times10^4 \text{ lb}_m/\text{ft}\cdot\text{s}^2$$

Here Appendix E has been used for the conversion of units. With the above information, Eq. 2.4-18 gives

$$Q = \frac{w}{\rho} = \frac{\pi\left(2.497\times10^4\right)(1.1/12)}{8\left(3.80\times10^{-2}\right)(27)}\left[\left[1-(0.45)^4\right] - \frac{\left[1-(0.45)^2\right]^2}{\ln(1/0.45)}\right]$$
$$= (0.6748)\left[(1-0.04101) - \frac{(1-0.2025)^2}{\ln(1/0.495)}\right]$$
$$= (0.6748)[0.1625] = 0.110 \text{ ft}^3/\text{s} \qquad \text{(A)}$$

To verify that the flow is indeed laminar, we next calculate the Reynolds number and get

$$\text{Re} = \frac{2R(1-\kappa)\langle v_z\rangle\rho}{\mu} = \frac{2w}{\pi R\mu(1+\kappa)}$$
$$= \frac{2\left(0.110 \text{ ft}^3/\text{s}\right)\left(80.3 \text{ lb}_m/\text{ft}^3\right)}{\pi(1.1/12 \text{ ft})\left(3{,}80\times10^{-2} \text{ lb}_m/\text{ft}\cdot\text{s}\right)(1.4)} = 1110 \qquad \text{(B)}$$

Since this value is well within the laminar range, our assumption of laminar flow is confirmed.

2A.4 Loss of catalyst particles in stack gas

(a) Rearrangement of Eq. 2.7-17 gives the terminal velocity as

$$v_t = \frac{D^2(\rho_s - \rho)g}{18\mu} \qquad \text{(A)}$$

in which D is the sphere diameter. Particles settling at v_t greater than the centerline gas velocity will not go up the stack. Hence, the value of D that corresponds to $v_t = 1.0$ ft/s will be the maximum diameter of particles that can be lost in the stack gas of the system under consideration.

Conversion of the data to cgs units gives

$$v_t = (1\ \text{ft/s})(12 \times 2.54\ \text{cm/ft}) = 30.48\ \text{cm/s}$$

$$\rho = \left(0.045\ \text{lb}_m/\text{ft}^3\right)\left(453.59\ \text{g/lb}_m\right)\left((12 \times 2.54\ \text{cm/ft})^{-3}\right)$$

$$= 7.2 \times 10^{-4}\ \text{g/cm}^3$$

Hence, the maximum allowable diameter is

$$D_{\max} = \sqrt{\frac{18\mu v_t}{(\rho_s - \rho)g}} = \sqrt{\frac{18(0.000026\ \text{g/cm}\cdot\text{s})(30.48\ \text{cm/s})}{\left(1.2 - 7.2 \times 10^{-4}\ \text{g/cm}^3\right)\left(980.7\ \text{cm/s}^2\right)}}$$

$$= 0.011\ \text{cm} = 110\ \text{microns} \qquad \text{(A)}$$

(b) Equation 2.7-17 was derived for Re <<1, but holds approximately up to Re = 1. For the problem discussed here

$$\text{Re} = \frac{Dv_t\rho}{\mu} = \frac{(0.011)(30.48)\left(7.2 \times 10^{-4}\right)}{0.00026} = 0.93 \qquad \text{(B)}$$

Therefore, the result in **(a)** is approximately correct. Methods are given in Chapter 6 for solving problems of this type outside the creeping-flow region.

2B.1 Simple shear flow between parallel plates
(a) For plates of length L and width W, a momentum balance is

$$LW\,\tau_{yx}\big|_y - LW\,\tau_{yx}\big|_{y+\Delta y} = 0 \tag{A}$$

Division by Δy and then letting Δy approach zero gives

$$\frac{d\tau_{yx}}{dy} = 0 \tag{B}$$

Integration with respect to y gives

$$\tau_{yx} = C_1 \tag{C}$$

Insertion of Newton's law of viscosity then leads to

$$-\mu\frac{dv_x}{dy} = C_1 \tag{D}$$

Integration of this equation gives

$$v_x = -\frac{C_1}{\mu}y + C_2 \tag{E}$$

Application of the boundary condition at $y = 0$, that $v_x = 0$, tells us that $C_2 = 0$. Then the boundary condition at $y = b$, that $v_x = v_0$, tells us that $C_1 = -\mu v_0/b$. Therefore the velocity and shear stress distributions are

$$v_x = \frac{v_0}{b}y \qquad \text{and} \qquad \tau_{yx} = -\mu\frac{v_0}{b} \tag{F}$$

(b) The volume rate flow through the slit is obtained by integrating the velocity over the cross section

$$Q=\int_0^W\int_0^b v_x(y)dydz=W\int_0^b v_x(y)dy=W\frac{v_0}{b}\int_0^b ydy=\tfrac{1}{2}Wbv_0 \qquad \text{(G)}$$

This seems like a reasonable result, inasmuch as it is the cross sectional area *Wb* multiplied by the average velocity through the cross section $\frac{1}{2}v_0$.

(c) Equations A through E still apply, but now the boundary conditions are different: the boundary condition at $y = 0$, that $v_x = v_0$, tells us that $C_2 = v_0$; and the boundary condition at $y = b$, that $v_x = 0$, tells us that $C_1 = \mu v_0/b$. Therefore the velocity and shear stress distributions are:

$$v_x = v_0\left(1-\frac{y}{b}\right) \quad \text{and} \quad \tau_{yx} = +\mu\frac{v_0}{b} \qquad \text{(H)}$$

The volumetric flow rate is the same as that given by Eq. G.

2B.2 Different choice of coordinates for the falling film problem

Set up a momentum balance as before, and obtain the differential equation

$$\frac{d\tau_{\bar{x}z}}{d\bar{x}} = \rho g \cos\beta \tag{A}$$

Integration gives

$$\tau_{\bar{x}z}(\bar{x}) = \rho g \bar{x} \cos\beta + C_1 \tag{B}$$

Since no momentum is transferred at $\bar{x} = \delta$, at that plane $\tau_{\bar{x}z} = 0$. This boundary condition enables us to find that $C_1 = -\rho g \delta \cos\beta$, and the momentum flux distribution is

$$\tau_{\bar{x}z}(\bar{x}) = -\rho g \delta \cos\beta \left(1 - \frac{\bar{x}}{\delta}\right) \tag{C}$$

Note that the momentum flux is in the negative $\bar{x}$-direction.

Insertion of Newton's law of viscosity $\tau_{\bar{x}z} = -\mu\left(dv_z/d\bar{x}\right)$ into the foregoing equation gives the differential equation for the velocity distribution:

$$\frac{dv_z}{d\bar{x}} = \left(\frac{\rho g \delta \cos\beta}{\mu}\right)\left(1 - \frac{\bar{x}}{\delta}\right) \tag{D}$$

This first-order differential equation can be integrated to give

$$v_z(\bar{x}) = \left(\frac{\rho g \delta \cos\beta}{\mu}\right)\left(\bar{x} - \frac{1}{\delta}\frac{\bar{x}^2}{2}\right) + C_2 = \left(\frac{\rho g \delta^2 \cos\beta}{\mu}\right)\left(\frac{\bar{x}}{\delta} - \frac{1}{2}\frac{\bar{x}^2}{\delta^2}\right) + C_2 \tag{E}$$

The constant C_2 is zero, because $v_z = 0$ at $\bar{x} = 0$. Therefore

$$v_z(\bar{x}) = \left(\frac{\rho g \delta^2 \cos\beta}{\mu}\right)\left[\frac{\bar{x}}{\delta} - \frac{1}{2}\left(\frac{\bar{x}}{\delta}\right)^2\right] \qquad \text{(F)}$$

We note that $\bar{x}$ and x are related by $\bar{x}/\delta = 1-(x/\delta)$. When this is substituted into the velocity distribution above, we get

$$v_z(x) = \left(\frac{\rho g \delta^2 \cos\beta}{\mu}\right)\left(\left(1-\frac{x}{\delta}\right) - \frac{1}{2}\left[1-2\frac{x}{\delta}+\left(\frac{x}{\delta}\right)^2\right]\right) \qquad \text{(G)}$$

which can be rearranged to give Eq. 2.2-22.

2B.3 Alternate procedure for solving flow problems

Substituting Eq. 2.2-18 into Eq. 2.2-14 gives

$$\frac{d}{dx}\left(-\mu\frac{dv_z}{dx}\right)=\rho g\cos\beta \qquad \text{or} \qquad \frac{d^2v_z}{dx^2}=-\frac{\rho g\cos\beta}{\mu} \tag{A}$$

Integrate twice with respect to x (see Eq. C.1-10) and get

$$v_z=-\frac{\rho g\cos\beta}{2\mu}x^2+C_1x+C_2 \tag{B}$$

Then use the no-slip boundary condition that $v_z=0$ at $x=\delta$, and the zero momentum flux boundary condition that $dv_z/dx=0$ at $x=0$. The second gives $C_1=0$, and the first gives $C_2=(\rho g\cos\beta/2\mu)\delta^2$. Substitution of these constants into Eq. B gives

$$v_z=-\frac{\rho g\cos\beta}{2\mu}x^2+\frac{\rho g\cos\beta}{2\mu}\delta^2 \tag{C}$$

This may be rearranged to give Eq. 2.2-22.

2B.4 Laminar flow in a narrow slit

(a) The momentum balance in §2.2 leads to

$$\frac{d}{dx}\tau_{xz} = \frac{(\mathscr{P}_0 - \mathscr{P}_L)}{L} \tag{A}$$

Integration gives

$$\tau_{xz}(x) = \frac{(\mathscr{P}_0 - \mathscr{P}_L)}{L}x + C_1 \tag{B}$$

Substitution of Newton's law , $\tau_{xz} = -\mu(dv_z/dx)$, into the above gives

$$\frac{dv_z}{dx} = -\frac{(\mathscr{P}_0 - \mathscr{P}_L)}{\mu L}x + \frac{C_1}{\mu} \tag{C}$$

Integration then gives

$$v_z(x) = -\frac{(\mathscr{P}_0 - \mathscr{P}_L)x^2}{2\mu L} + \frac{C_1}{\mu}x + C_2 \tag{D}$$

Use of the no-slip boundary conditions at $x = \pm B$ gives C_1 **=** 0 and $C_2 = (\mathscr{P}_0 - \mathscr{P}_L)B^2/2\mu L$. Substitution of these constants into Eqs. B and D gives the stress and velocity distributions as

$$\tau_{xz}(x) = \frac{(\mathscr{P}_0 - \mathscr{P}_L)}{L}x \tag{E}$$

$$v_z(x) = \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{2\mu L}\left[1 - \left(\frac{x}{B}\right)^2\right] \tag{F}$$

(b) The maximum velocity is at the middle of the slit and is

$$v_{z,\max} = \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{2\mu L} \tag{G}$$

and hence

$$\frac{v_z(x)}{v_{z,\max}} = 1 - \left(\frac{x}{B}\right)^2 \tag{H}$$

The ratio of the average to the maximum velocity is then

$$\frac{\langle v_z \rangle}{v_{z,\max}} = \frac{\int_0^W \int_{-B}^{B} \left[1 - (x/B)^2\right] dx dy}{\int_0^W \int_{-B}^{B} dx dy} = \frac{\int_0^1 (1 - \xi^2) d\xi}{\int_0^1 d\xi} = \left(1 - \tfrac{1}{3}\right) = \tfrac{2}{3} \tag{I}$$

(c) The mass rate of flow is

$$w = \rho(2BW)\langle v_z \rangle = \rho(2BW)\left(\tfrac{2}{3}\right)\frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{2\mu L} = \frac{2}{3}\frac{(\mathscr{P}_0 - \mathscr{P}_L)\rho B^3 W}{\mu L} \tag{J}$$

The volume rate of flow is

$$Q = \frac{w}{\rho} = \frac{2}{3}\frac{(\mathscr{P}_0 - \mathscr{P}_L)B^3 W}{\mu L} \tag{K}$$

(e) This problem is equivalent to that described in §2.5 if we set both viscosities equal to μ, and set b equal to B. Then the maximum velocity is given by the prefactor (e.g., the term outside of the square brackets) in either Eq. 2.5-18 or Eq. 2.5-19, which is equivalent to that given by Eq. G above. The average velocity is then given by Eq. 2.5-20 or Eq. 2.5-21, and the resulting ratio of the average and maximum velocity is equivalent to Eq. I above.

2B.5 Laminar slit flow with a moving wall ("plane Couette flow")

(a) Start with the velocity distribution from part (*a*) of Problem 2B.4 (in terms of the integration constants).

$$v_z(x) = -\frac{(\mathscr{P}_0 - \mathscr{P}_L)x^2}{2\mu L} + \frac{C_1}{\mu}x + C_2 \tag{A}$$

Determine C_1 and C_2 from the boundary conditions that $v_z = 0$ at $x = -B$, and $v_z = v_0$ at $x = +B$. This leads to

$$0 = -\frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{2\mu L} - \frac{C_1}{\mu}B + C_2 \qquad \text{(for } x = -B\text{)} \tag{B}$$

$$v_0 = -\frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{2\mu L} + \frac{C_1}{\mu}B + C_2 \qquad \text{(for } x = +B\text{)} \tag{C}$$

We now have two simultaneous equations that have to be solved for the integration constants, C_1 and C_2. Addition of Eqs. B and C and rearranging gives

$$C_2 = \frac{v_0}{2} + \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{2\mu L} \tag{D}$$

Then subtracting Eq. B from Eq. C and rearranging gives

$$C_1 = \frac{v_0}{2}\frac{\mu}{B} \tag{E}$$

Putting these values for C_1 and C_2 into Eq. A gives finally

$$v_z(x) = \frac{(\mathscr{P}_0 - \mathscr{P})B^2}{2\mu L}\left[1 - \left(\frac{x}{B}\right)^2\right] + \frac{v_0}{2}\left(1 + \frac{x}{B}\right) \tag{F}$$

Notice that the velocity distribution is no longer symmetric about the midplane, so that $C_1 \neq 0$.

Eq. F can be differentiated with respect to x, and then Newton's law of viscosity, $\tau_{xz} = -\mu\left(dv_z/dx\right)$, can be used to get

$$\tau_{xz}(x) = -\mu\left[\left(\frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{2\mu L}\right)\left(-\frac{2x}{B^2}\right) + \frac{v_0}{2B}\right] = \left(\frac{\mathscr{P}_0 - \mathscr{P}_L}{L}\right)x - \frac{\mu v_0}{2B} \qquad \text{(G)}$$

for the shear stress distribution.

(b) The velocity distribution is given by Eq. F above.

2B.6 Interrelation of slit and annulus formulas

From the annular flow result in Eq. 2.4-18 we get, by replacing κ by $1-\varepsilon$

$$w=\left(\frac{\pi(\mathscr{P}_0-\mathscr{P}_L)R^4\rho}{8\mu L}\right)\left[\left(1-(1-\varepsilon)^4\right)+\frac{\left(1-(1-\varepsilon)^2\right)^2}{\ln(1-\varepsilon)}\right] \tag{A}$$

Now we are mainly concerned with the expansion of the expression in the bracket on the right side of Eq. A, which we abbreviate as [ ].

$$\begin{aligned}
[\ \ ]&=\left(1-1+4\varepsilon-6\varepsilon^2+4\varepsilon^3-\varepsilon^4\right)+\frac{\left(1-1+2\varepsilon-\varepsilon^2\right)^2}{-\varepsilon-\frac{1}{2}\varepsilon^2-\frac{1}{3}\varepsilon^3-\frac{1}{4}\varepsilon^4-\cdots}\\
&=\left(4\varepsilon-6\varepsilon^2+4\varepsilon^3-\varepsilon^4\right)-\frac{\left(4\varepsilon^2-4\varepsilon^3+\varepsilon^4\right)}{\varepsilon+\frac{1}{2}\varepsilon^2+\frac{1}{3}\varepsilon^3+\frac{1}{4}\varepsilon^4+\cdots}\\
&=\left(4\varepsilon-6\varepsilon^2+4\varepsilon^3-\varepsilon^4\right)-\left(4\varepsilon-6\varepsilon^2+\frac{8}{3}\varepsilon^3-\frac{1}{3}\varepsilon^4+\cdots\right)\\
&=\left(4\varepsilon^3-\varepsilon^4\right)-\left(\frac{8}{3}\varepsilon^3-\frac{1}{3}\varepsilon^4+\cdots\right)\approx\frac{4}{3}\varepsilon^3\left(1-\frac{1}{2}\varepsilon\right)\quad\text{for small }\varepsilon
\end{aligned} \tag{B}$$

Hence for small ε, Eq. A becomes

$$w=\left(\frac{\pi(\mathscr{P}_0-\mathscr{P}_L)R^4\rho}{8\mu L}\right)\frac{4}{3}\varepsilon^3\left(1-\frac{1}{2}\varepsilon\right)=\frac{\pi(\mathscr{P}_0-\mathscr{P}_L)R^4\varepsilon^3\rho}{6\mu L}\left(1-\frac{1}{2}\varepsilon\right) \tag{C}$$

This gives, finally, a result in agreement with Eq. 2B.6-1, which was obtained by modifying the slit formula.

2B.7 Flow of a film on the outside of a circular tube

(a) The momentum balance is the same as that in §2.3, leading to Eq. 2.3-12. When the pressure-difference term is omitted, because the film is moving solely to the force of gravity, we get

$$-\frac{d}{dr}(r\tau_{rz})+\rho g r=0 \tag{A}$$

When Newton's law of viscosity is inserted, the equation becomes

$$\mu\frac{d}{dr}\left(r\frac{dv_r}{dr}\right)+\rho g r=0 \tag{B}$$

Integration then gives

$$v_z(r)=-\frac{\rho g r^2}{4\mu}+C_1\ln r+C_2 \tag{C}$$

The constants of integration are determined from the boundary conditions that at $r = R$, $v_z=0$ (zero slip at the solid surface) and that at $r = aR$, $dv_z/dr=0$ (no radial momentum transport at the free surface). The constants of integration are $C_1=\rho g(aR)^2/2\mu$ and $C_2=\left(\rho g R^2/4\mu\right)+\left(\rho g(aR)^2/2\mu\right)$. When these constants are put into Eq. C, we get finally

$$v_z(r)=\frac{\rho g R^2}{4\mu}\left[1-\left(\frac{r}{R}\right)^2+2a^2\ln\left(\frac{r}{R}\right)\right] \tag{D}$$

(b) The mass rate of flow is $\rho v_z(r)$ integrated over the cross-section of the flow

$$w=\int_0^{2\pi}\int_R^{aR}\rho v_z(r)rdrd\theta\ =2\pi\rho\int_R^{aR}v_z(r)rdr \tag{E}$$

In the second expression, we have taken the density outside of the integral, assuming that the fluid is incompressible. We have also performed the integration over the angular variable. It is a little easier to continue if we use the dimensionless variable $\xi = r/R$. Then Eq. E becomes

$$2\pi\rho R^2 \int_1^a v_z(\xi)\xi d\xi \tag{F}$$

We next insert the expression for the velocity from Eq. D, but written in terms of the dimensionless variable of integration.

$$w = 2\pi\rho R^2 \int_1^a \frac{\rho g R^2}{4\mu}\left(1 - \xi^2 + 2a^2 \ln \xi\right) d\xi \tag{G}$$

When the integration is performed, we get

$$\begin{aligned} w &= \frac{\pi\rho^2 g R^2}{2\mu}\left[\tfrac{1}{2}\xi^2 - \tfrac{1}{4}\xi^4 + 2a^2\left(-\tfrac{1}{4}\xi^2 + \tfrac{1}{2}\xi^2 \ln \xi\right)\right]\Bigg|_1^a \\ &= \frac{\pi\rho^2 g R^2}{8\mu}\left(-1 + 4a^2 - 3a^4 + 4a^4 \ln a\right) \end{aligned} \tag{H}$$

(c) If we set $a = 1 + \varepsilon$ (where ε is small), and expand in powers of ε, we get

$$w = \frac{\pi\rho^2 g R^4}{8\mu}\begin{bmatrix} -1 + 4\left(1 + 2\varepsilon + \varepsilon^2\right) - 3\left(1 + 4\varepsilon + 6\varepsilon^2 + 4\varepsilon^3 + \varepsilon^4\right) \\ +4\left(1 + 4\varepsilon + 6\varepsilon^2 + 4\varepsilon^3 + \varepsilon^4\right)\left(\varepsilon - \tfrac{1}{2}\varepsilon^2 + \tfrac{1}{3}\varepsilon^3 - \tfrac{1}{4}\varepsilon^4 + \cdots\right) \end{bmatrix} \tag{I}$$

Here we have use the Taylor series expansion for $\ln(1+\varepsilon)$ in Eq. C.2-3. We find that the terms without ε sum to zero, as do the terms in ε and ε^2. Thus we get

$$w = \frac{\pi\rho^2 g R^4}{8\mu}\left[\tfrac{16}{3}\varepsilon^3 + O\left(\varepsilon^4\right)\right] \approx \frac{2\pi\rho^2 g R^4 \varepsilon^3}{3\mu} \tag{J}$$

Now, does this simplify to the result in Eq. 2.2-26? If we make the identification $W = 2\pi R$ for the width of the film, and $\delta = \varepsilon R$ for the film thickness, then Eq. J is indeed equivalent to Eq. 2.2-26 (provided that $\cos\beta = 1$, that is, the film is on a vertical surface).

2B.8 Annular flow with inner cylinder moving axially

(a) The momentum balance is the same as that in Eq. 2.4-2, but with the pressure-difference term omitted. We can substitute Newton's law of viscosity into this equation to get

$$-\mu\frac{dv_z}{dr}=\frac{C_1}{r} \tag{A}$$

whence

$$v_z(r)=-\frac{C_1}{\mu}\ln r+C_2 \qquad \text{or} \qquad \frac{v_z(r)}{v_0}=-D_1\ln\frac{r}{R}+D_2 \tag{B}$$

That is, we rewrite Eq. B in such a way that only dimensionless quantities appear, including the new integration constants D_1 and D_2. These integration constants are determined from the no-slip conditions at the cylindrical surfaces: $v_z(\kappa R)=v_0$ and $v_z(R)=0$. The constants of integration are $D_2=0$ and $D_1=-1/\ln\kappa$ and the velocity distribution

$$\frac{v_z}{v_0}=\frac{\ln(r/R)}{\ln\kappa} \tag{C}$$

(b) The mass rate of flow w is obtained by integrating the mass flux ρv_z over the cross section of flow

$$w=\int_0^{2\pi}\int_{\kappa R}^{R}\rho v_z r dr d\theta=2\pi\rho\frac{v_0R^2}{\ln\kappa}\int_\kappa^1(\ln\xi)\xi d\xi=2\pi\rho\frac{v_0R^2}{ln\,\kappa}\left(\tfrac{1}{2}\xi^2\ln\xi-\tfrac{1}{4}\xi^2\right)\Big|_\kappa^1$$

$$=2\pi\rho\frac{v_0R^2}{\ln\kappa}\left[-\tfrac{1}{2}\kappa^2\ln\kappa-\tfrac{1}{4}\left(1-\kappa^2\right)\right]=\pi R^2\rho v_0\left[\frac{\left(1-\kappa^2\right)}{2\ln(1/\kappa)}-\kappa^2\right] \tag{D}$$

(c) The force on a length L of the rod is

$$F_z = \int_0^L \int_0^{2\pi} \left(-\tau_{rz}\right)\Big|_{r=\kappa R} \kappa R d\theta dz = \int_0^L \int_0^{2\pi} \left(+\mu \frac{dv_z}{dr}\right)\Bigg|_{r=\kappa R} \kappa R d\theta dz$$

$$= 2\pi\kappa R L \mu v_0 \frac{(1/\kappa R)}{\ln \kappa} = -\frac{2\pi L \mu v_0}{\ln(1/\kappa)} \qquad \text{(E)}$$

which is a force in the direction opposite to the direction of flow.

(d) When we replace κ by $1-\varepsilon$ and expand $\ln(1-\varepsilon)$ in a Taylor series, we get

$$F_z = \frac{-2\pi L \mu v_0}{\ln 1 - \ln(1-\varepsilon)} =$$

$$= -2\pi L \mu v_0 \frac{1}{+\left(\varepsilon + \tfrac{1}{2}\varepsilon^2 + \tfrac{1}{3}\varepsilon^3 + \tfrac{1}{4}\varepsilon^4 + \cdots\right)}$$

$$= \frac{-2\pi L \mu v_0}{\varepsilon} \frac{1}{\left(1 + \tfrac{1}{2}\varepsilon + \tfrac{1}{3}\varepsilon^2 + \tfrac{1}{4}\varepsilon^3 + \cdots\right)}$$

$$= -\frac{2\pi L \mu v_0}{\varepsilon}\left(1 - \tfrac{1}{2}\varepsilon - \tfrac{1}{12}\varepsilon^2 + \cdots\right) \qquad \text{(F)}$$

To get this last result, one can expand the fraction $1/(1+\delta)$, where $\delta = \tfrac{1}{2}\varepsilon + \tfrac{1}{3}\varepsilon^2 + \tfrac{1}{4}\varepsilon^3 + \cdots$, in a Taylor series about $\delta = 0$ using Eq. C.2-1.

2B.9 Analysis of a capillary flowmeter

Designate the water by fluid "I" and the carbon tetrachloride by "II". Label the distance from B to C as "J". The mass rate of flow in the tube section "AB" is given by

$$w = \frac{\pi\left(\mathscr{P}_A - \mathscr{P}_B\right)R^4\rho_{\mathrm{I}}}{8\mu L} = \frac{\pi\left[\left(p_A - p_B\right) + \rho_{\mathrm{I}} g h\right]R^4\rho_{\mathrm{I}}}{8\mu L} \tag{A}$$

Since the fluid in the manometer is not moving, the pressures at D and E must be equal; hence

$$p_A + \rho_{\mathrm{I}} g h + \rho_{\mathrm{I}} g\left(J + H\right) = p_B + \rho_{\mathrm{I}} g J + \rho_{\mathrm{II}} g H \tag{B}$$

from which we get

$$p_A - p_B + \rho_{\mathrm{I}} g h = \left(\rho_{\mathrm{II}} - \rho_{\mathrm{I}}\right) g H \tag{C}$$

Insertion of this into the first equation above gives the expression for the mass rate of flow in terms of the difference in the densities of the two fluids, the acceleration of gravity, and the height H

$$w = \frac{\pi\left[\left(\rho_{\mathrm{II}} - \rho_{\mathrm{I}}\right) g H\right]R^4\rho_{\mathrm{I}}}{8\mu L} \tag{D}$$

This verifies that θ need not be measured.

Using the values of $\mu = 1.0019\times10^{-3}\ \mathrm{Pa\cdot s}$ and $\rho_{\mathrm{I}} = 0.998\ \mathrm{g/cm^3}$ from Table 1.4-1, along with the other parameter values in the problem statement, the mass flow rate is

$$w = \frac{\pi\left[\left(1.594 - 0.998\ \mathrm{g/cm^3}\right)\left(9.8\ \mathrm{m/s^2}\right)\left(1\ \mathrm{in.}\right)\right]\left(0.01\ \mathrm{in.}\right)^4\left(0.998\ \mathrm{g/cm^3}\right)}{8\left(1.0019\times10^{-3}\ \mathrm{Pa\cdot s}\right)\left(120\ \mathrm{in.}\right)}$$

$$\times\left(\frac{0.0254\ \mathrm{m}}{\mathrm{in.}}\right)^4\left(\frac{100\ \mathrm{cm}}{\mathrm{m}}\right)^6\left(\frac{1\ \mathrm{kg}}{1000\ \mathrm{g}}\right)^2\left(\frac{1\ \mathrm{Pa\cdot m^2}}{1\ \mathrm{N}}\right)\left(\frac{1\ \mathrm{N\cdot s^2}}{1\ \mathrm{kg\cdot m}}\right)$$

$$= 7.925 \times 10^{-8} \text{ kg/s} \left(\frac{2.2046 \text{ lb}_m}{\text{kg}} \right) \left(\frac{3600 \text{ s}}{\text{hr}} \right)$$

$$= 6.29 \times 10^{-4} \text{ lb}_m/\text{hr} \qquad \text{(E)}$$