Solutions to Problems in Chapter 3

3.1)

We have $x(t) = u(t+1) - 2u(t) + u(t-1)$, where $u(t)$ is a unit step function. Using basic operations on signals, we have:

a) $g(t) = x(t) + x(t-1) + x(-t) = u(t+1) - 2u(t) + u(t-1) + u(t) - 2u(t-1) + u(t-2) + u(-t+1) - 2u(-t) + u(-t-1) = x(t-1)$

b) $g(t) = \frac{dx(t)}{dt} = \delta(t+1) - 2\delta(t) + \delta(t-1)$

Note that $\delta(t)$ is a unit impulse function.

c) $g(t) = \int x(t)dt \; = -|\text{t}| + 1$

d) $g(t) = x(2t) = u(2t+1) - 2u(2t+1) + u(2t-1)$

e) $g(t) = x(2t-1) = u(2t) - 2u(2t-1) + u(2t-2)$

Note that in part e), the precedence rule should be followed.

3.2)

First of all, it is important to note that the differentiation and integration operations are both linear, and a trigonometric function is a nonlinear function of its argument. The first three systems are all linear since they satisfy the linearity requirement, as stated in (3.14). The reason lies in the fact that in all these three systems, the amplitude of the sinusoidal signal consists of the input or its integral or its derivative, whereas in the second three systems, it is the argument of the sinusoidal signal which has the input or its integral or its derivative, and thus do not meet the linearity requirement.

3.3)

a) $f(t) = \sin(6\pi t) + \cos(4\pi t)$. The period of a sum signal is the least common multiple of the periods of the two signals. The period of $\sin(6\pi t)$ is $\frac{1}{3}$ and the period of $\cos(4\pi t)$ is $\frac{1}{2}$. The period of $f(t)$ is thus 1, as the least common multiple of $\frac{1}{3}$ and $\frac{1}{2}$ is 1.

b) $f(t) = 1 + \cos(t)$. 1 is a constant and does not impact the period of $f(t)$. Since the period of $\cos(t)$ is 2π, the period of $f(t)$ is 2π.

c) $f(t) = \exp(-|t|)\cos(2\pi t)$. One of the requirements for a product signal to be periodic is that both signals be periodic. Since $\exp(-|t|)$ is not periodic, $f(t)$ is not periodic.

d) $f(t) = \big(u(t) - u(t-1)\big)\cos(7\pi t)$. One of the requirements for a product signal to be periodic is that neither signals be time-limited. Since $u(t) - u(t-1)$ is time-limited, $f(t)$ is not periodic.

e) $f(t) = \sin(\sqrt{2}\pi t) + \cos(2t)$. The period of a sum signal is the least common multiple of the periods of the two signals. The period of $\sin(\sqrt{2}\pi t)$ is $\sqrt{2}$ and the period of $\cos(2t)$ is π. The least common multiple of these two irrational numbers does not exist, so $f(t)$ is not periodic.

3.4)

In this problem, we use the results given in Table 3.1

a) It is a power signal, as we have:

$$P = \lim_{S\to\infty}\left(\frac{1}{2S}\right)\int_{-S}^{S}|g(t)|^2 dt$$

$$= \lim_{S\to\infty}\left(\frac{1}{2S}\right)\int_{-S}^{S}\left|A\exp\big(j(2\pi t+\theta)\big)\right|^2 dt = \lim_{S\to\infty}\left(\frac{1}{2S}\right)\int_{-S}^{S}A^2 dt = A^2 < \infty$$

b) It is a power signal, as we have:

$$P = \lim_{S\to\infty}\left(\frac{1}{2S}\right)\int_{-S}^{S}|g(t)|^2dt = \lim_{S\to\infty}\left(\frac{1}{2S}\right)\int_{-S}^{S}|\mathrm{sgn}(t)|^2dt = \lim_{S\to\infty}\left(\frac{1}{2S}\right)\int_{-S}^{S}dt = 1 < \infty$$

c) It is an energy signal, as we have:

$$E = \int_{-\infty}^{\infty}|g(t)|^2dt = \int_{-\infty}^{\infty}|\exp(-2|t|)|^2dt = \int_{-\infty}^{\infty}\exp(-4|t|)\,dt = 2\int_{0}^{\infty}\exp(-4t)\,dt = \frac{1}{2} < \infty$$

d) It is neither an energy signal nor a power signal, as we have:

$$E = \int_{-\infty}^{\infty}|g(t)|^2dt = \int_{-\infty}^{\infty}|tu(t)|^2dt = \int_{0}^{\infty}t^2dt = \infty$$

$$P = \lim_{S\to\infty}\left(\frac{1}{2S}\right)\int_{-S}^{S}|g(t)|^2dt = \lim_{S\to\infty}\left(\frac{1}{2S}\right)\int_{-S}^{S}|tu(t)|^2dt = \lim_{S\to\infty}\left(\frac{1}{2S}\right)\int_{0}^{\infty}t^2dt = \infty$$

e) Since it is a periodic signal, it is a power signal. With a period of 2π, we have:

$$P = \left(\frac{1}{T_0}\right)\int_{-T_0/2}^{T_0/2}|g(t)|^2dt = \left(\frac{1}{2\pi}\right)\int_{-\pi}^{\pi}|1+\cos(t)|^2dt = \frac{3}{2} < \infty$$

3.5)

In this problem, we use (3.14) to check the linearity property and (3.15) to check the time-invariance property.

a) It is a linear and time-invariant system.

b) It is a linear and time-varying system.

c) It is a non-linear and time-varying system.

d) It is a non-linear and time-invariant system.

e) It is a linear and time-varying system.

3.6)

a) $g(t) = \cos(t) + \sin(t) + \sin(t)\cos(t)$

We have $g(t) \neq g(-t)$ and also $g(t) \neq -g(-t)$, it is thus neither an even signal nor an odd signal, and we need to find the even and odd parts of $g(t)$. Using (3.12), we get:

$$g_e(t) = 0.5\big(g(t) + g(-t)\big)$$
$$= 0.5(\cos(t) + \sin(t) + \sin(t)\cos(t) + \cos(-t) + \sin(-t) + \sin(-t)\cos(-t)) = \cos(t)$$

and

$$g_o(t) = 0.5\big(g(t) - g(-t)\big)$$
$$= 0.5(\cos(t) + \sin(t) + \sin(t)\cos(t) - \cos(-t) - \sin(-t) - \sin(-t)\cos(-t)) = \sin(t) + \sin(t)\cos(t)$$

b) $g(t) = tu(t)$

We have $g(t) \neq g(-t)$ and also $g(t) \neq -g(-t)$, it is thus neither an even signal nor an odd signal, and we need to find the even and odd parts of $g(t)$. Using (3.12), we get:

$$g_e(t) = 0.5(g(t) + g(-t)) = 0.5(tu(t) - tu(-t)) = 0.5|t|$$

and

$$g_o(t) = 0.5(g(t) - g(-t)) = 0.5(tu(t) + tu(-t)) = 0.5t$$

c) $g(t) = \sin(4\pi t + \pi/5)$

We have $g(t) \neq g(-t)$ and also $g(t) \neq -g(-t)$, it is thus neither an even signal nor an odd signal, and we need to find the even and odd parts of $g(t)$. Using (3.12), we get:

$$\begin{aligned} g_e(t) &= 0.5(g(t) + g(-t)) = 0.5(\sin(4\pi t + \pi/5) + \sin(-4\pi t + \pi/5)) \\ &= \sin(\pi/5)\cos(4\pi t) \end{aligned}$$

and

$$\begin{aligned} g_o(t) &= 0.5(g(t) - g(-t)) = 0.5(\sin(4\pi t + \pi/5) - \sin(-4\pi t + \pi/5)) \\ &= \cos(\pi/5)\sin(4\pi t) \end{aligned}$$

d) $g(t) = \exp(-|t|)\sin(t)$. It is an odd signal, as $g(t) = -g(-t)$.

e) $g(t) = u(t+1) - u(t-1)$. It is an even signal, as $g(t) = g(-t)$.

3.7)

a) It is a non-causal system with memory.

b) It is a causal system with no memory.

c) It is a causal system with no memory.

d) It is a non-causal system with no memory.

e) It is a causal system with memory.

3.8)

a) $y(t) = x(t)$

b) $y(t) = x(t+1)$

c) $y(t) = x(t)\cos(t)$

d) $y(t) = x(t+4)\cos(2t)$

e) $y(t) = x^2(t)$

f) $y(t) = x^3(t+9)$

g) $y(t) = x^{0.2}(t)\cos(4t)$

h) $y(t) = x^{-2}(t+1.5)\cos(6t)$

3.9)

a) The signals $y(t)$ and $z(t)$ are neither even nor odd. The signal $y(t)$ can be thus decomposed into an even part $y_e(t)$ and an odd part $y_o(t)$, i.e., we have:

$$y(t) = y_e(t) + y_o(t)$$

We then define $z(t)$ in terms of $y(t)$ as follows:

$$z(t) = y(-t) = y_e(-t) + y_o(-t) = y_e(t) - y_o(t)$$

Therefore, we have

$$\begin{aligned} y(t)z(t) &= \big(y_e(t) + y_o(t)\big)\big(y_e(t) - y_o(t)\big) \\ &= y_e(t)y_e(t) - y_e(t)y_o(t) + y_o(t)y_e(t) - y_o(t)y_o(t) \\ &= y_e(t)y_e(t) - y_o(t)y_o(t) \end{aligned}$$

Since the product of two even signals is even, the product of two odd signals is also even, and the difference between two even signals is even, $y(t)z(t)$ is an even signal.

b) The signal $x(t) = \sqrt{2}\sin(t)$ has unit power as half of the square of the amplitude of this sinusoid is one and a periodic sine wave with initial phase of zero is also an odd function.

c) An even rectangular pulse $p(t)$ with duration T and height $\frac{1}{\sqrt{T}}$ has unit power.

$$p(t) = \frac{1}{\sqrt{T}}\left(u\left(t + \frac{T}{2}\right) - u\left(t - \frac{T}{2}\right)\right)$$

d) Suppose the even part is $\cos(t)$ function which is periodic and the odd part is t which is nonperiodic, we thus have the following;

$$y(t) = y_e(t) + y_o(t) = \cos(t) + t$$

3.10)

Using a trigonometric identity, we have

$$x(t) = \cos(t) + \sin(t) = \sqrt{2}\cos\left(t - \frac{\pi}{4}\right)$$

Since $x(t)$ is a sinusoidal function, it is periodic and its period T in seconds is as follows:

$$t = \frac{2\pi t}{T} \rightarrow T = 2\pi$$

The fundamental frequency f in Hertz is thus as follows:

$$f = \frac{1}{T} = \frac{1}{2\pi}$$

Since $x(t)$ is periodic, it is a power signal, and has the following finite power:

$$P = \left(\frac{1}{T_0}\right) \int_{-T_0/2}^{T_0/2} |x(t)|^2 dt = P = \left(\frac{1}{2\pi}\right) \int_{-\pi}^{\pi} \left(\sqrt{2}\cos\left(t - \frac{\pi}{4}\right)\right)^2 dt = 1$$

To determine the minimum and maximum values of $x(t)$, we must first differentiate it with respect to time t and then set it equal to zero, i.e., we have:

$$\frac{dx(\mathrm{t})}{dt} = \frac{d(\cos(t) + \sin(t))}{dt}$$
$$= -\sin(t) + \cos(t) = 0 \rightarrow \sin(t) = \cos(t) \rightarrow t = \frac{\pi}{4} + k\pi, \quad k = 0, 1, 2, 3 \ \dots$$

At $= \frac{\pi}{4} + k\pi, k = 0, 2, 4 \dots$, $x(t)$ has its maximum value, which is as follows:

$$x\left(\frac{\pi}{4} + k\pi\right) = \sqrt{2}$$

At $= \frac{\pi}{4} + k\pi, k = 1, 3, 5 \dots$, $x(t)$ has its maximum value, which is as follows:

$$x\left(\frac{\pi}{4} + k\pi\right) = -\sqrt{2}$$

3.11)

The total energy of an energy signal $g(t) = g_e(t) + g_0(t)$, where $g_e(t)$ and $g_0(t)$ are its even and odd parts, is as follows:

$$E = \int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} \left(g(t)\right)^2 dt = \int_{-\infty}^{\infty} \left(g_e(t) + g_0(t)\right)^2 dt$$

$$= \int_{-\infty}^{\infty} \left(g_e(t)\right)^2 dt + \int_{-\infty}^{\infty} \left(g_0(t)\right)^2 dt + 2 \int_{-\infty}^{\infty} g_e(t) g_0(t) dt$$

Note that since $g_e(t)$ is an even function and $g_o(t)$ is an odd function, their product is an odd function and since the limits of the definite integral are symmetric with respect to zero, the following term is zero:

$$\int_{-\infty}^{\infty} g_e(t) g_0(t) dt = 0$$

We therefore have

$$E = E_e + E_o$$

3.12)

Using (3.37b), we have

$$c_n = \left(\frac{1}{T_0}\right) \int_{-T_0/2}^{T_0/2} g(t) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt$$

$$= \left(\frac{1}{T_0}\right) \int_{-T_0/2}^{0} g(t) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt + \left(\frac{1}{T_0}\right) \int_{0}^{T_0/2} g(t) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt$$

Substituting t for $-t$ in the first integral, noting $g(t) = g(-t)$, and using (3.17), the Euler's identity, we have

$$c_n = \left(\frac{1}{T_0}\right)\int_0^{T_0/2} g(-t)\exp\left(\frac{j2\pi nt}{T_0}\right)dt + \left(\frac{1}{T_0}\right)\int_0^{T_0/2} g(t)\exp\left(-\frac{j2\pi nt}{T_0}\right)dt$$

$$= \left(\frac{1}{T_0}\right)\int_0^{T_0/2} g(t)\exp\left(\frac{j2\pi nt}{T_0}\right)dt + \left(\frac{1}{T_0}\right)\int_0^{T_0/2} g(t)\exp\left(-\frac{j2\pi nt}{T_0}\right)dt$$

$$= \left(\frac{2}{T_0}\right)\int_0^{T_0/2} g(t)\cos\left(\frac{2\pi nt}{T_0}\right)dt$$

c_n is thus a pure real function (i.e., it has no imaginary part) and since $\cos\left(\frac{2\pi nt}{T_0}\right)$ is an even function of n, c_n is also an even function of n. Using (3.37b), we have

$$c_n = \left(\frac{1}{T_0}\right)\int_{-T_0/2}^{T_0/2} g(t)\exp\left(-\frac{j2\pi nt}{T_0}\right)dt$$

$$= \left(\frac{1}{T_0}\right)\int_{-T_0/2}^{0} g(t)\exp\left(-\frac{j2\pi nt}{T_0}\right)dt + \left(\frac{1}{T_0}\right)\int_0^{T_0/2} g(t)\exp\left(-\frac{j2\pi nt}{T_0}\right)dt$$

Substituting t for $-t$ in the first integral, noting $g(t) = -g(-t)$, and using (3.17), we have

$$c_n = \left(\frac{1}{T_0}\right)\int_0^{T_0/2} g(-t)\exp\left(\frac{j2\pi nt}{T_0}\right)dt + \left(\frac{1}{T_0}\right)\int_0^{T_0/2} g(t)\exp\left(-\frac{j2\pi nt}{T_0}\right)dt$$

$$= \left(\frac{1}{T_0}\right)\int_0^{T_0/2} -g(t)\exp\left(\frac{j2\pi nt}{T_0}\right)dt + \left(\frac{1}{T_0}\right)\int_0^{T_0/2} g(t)\exp\left(-\frac{j2\pi nt}{T_0}\right)dt$$

$$= \left(-\frac{2j}{T_0}\right)\int_0^{T_0/2} g(t)\sin\left(\frac{2\pi nt}{T_0}\right)dt$$

c_n is thus a pure imaginary function (i.e., it has no real part) and since $\sin\left(\frac{2\pi nt}{T_0}\right)$ is an odd function of n, c_n is also an odd function of n.

3.13)

For a real-valued periodic $g(t)$, the complex exponential Fourier series is as follows:

$$c_n = \left(\frac{1}{T_0}\right)\int_{-T_0/2}^{T_0/2} g(t)\exp\left(\frac{-j2\pi nt}{T_0}\right)dt = \left(\frac{1}{T_0}\right)\int_{-T_0/2}^{T_0/2} g(t)\exp\left(\frac{+j2\pi(-n)t}{T_0}\right)dt = c_{-n}{}^*$$

where $c_n{}^*$ is the complex conjugate of c_n. We therefore have

$$|c_n| = |c_{-n}{}^*|$$

and

$$\measuredangle c_n = -\measuredangle c_n$$

For a real-valued periodic signal, the amplitude spectrum is an even function of n and the phase spectrum is an odd function of n.

3.14)

As reflected in Table 3.1, the average power of the periodic signal $g(t)$ is as follows:

$$P = \left(\frac{1}{T_0}\right)\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |g(t)|^2 dt = \left(\frac{1}{T_0}\right)\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} g(t)g^*(t)dt = \left(\frac{1}{T_0}\right)\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} g^*(t)\left(\sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{j2\pi nt}{T_0}\right)\right)dt$$

$$= \left(\frac{1}{T_0}\right)\sum_{n=-\infty}^{\infty} c_n \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} g^*(t)\exp\left(\frac{j2\pi nt}{T_0}\right)dt = \left(\frac{1}{T_0}\right)\sum_{n=-\infty}^{\infty} c_n(c_n{}^*T_0) = \sum_{n=-\infty}^{\infty} |c_n|^2$$

As we replaced $g(t)$ by the complex exponential form of its Fourier series, interchanged the order of summation and integration, and used the definition of c_n. The power of a periodic signal can thus be determined by the knowledge of the amplitude spectrum only. Note that in the calculation of power, phase plays no role.

3.15)

For the periodic signal $g(t) = \exp(t), 0 < t < 2\pi$ with $g(t + 2\pi) = g(t)$, the complex exponential Fourier series coefficients are as follows:

$$c_n = \left(\frac{1}{T_0}\right)\int_{-T_0/2}^{T_0/2} g(t)\exp\left(-\frac{j2\pi nt}{T_0}\right)dt = \left(\frac{1}{2\pi}\right)\int_{-\pi}^{\pi}\exp(t)\exp(-jnt)dt$$

$$= \left(\frac{1}{2\pi}\right)\int_{-\pi}^{\pi}\exp\big(t(1-jn)\big)\,dt =$$

$$= \left(\frac{1}{2\pi}\right)\left(\frac{1}{1-jn}\right)\Big(\exp\big(\pi(1-jn)\big) - \exp\big(\pi(-1+jn)\big)\Big)$$

$$= \frac{\cos(n\pi)(e^{\pi}-e^{-\pi})}{2\pi(1+n^2)}(1+jn))$$

3.16)

We could either use the set of equations (3.34a) to (3.34d) or we could use Euler's identity to get the following:

$$g(t) = 1 + \sin(2\pi t) + \cos^2(2\pi t)$$

$$= 1 + \left(\frac{\exp(j2\pi t) - \exp(-j2\pi t)}{2j}\right) + \left(\frac{\exp(j2\pi t) + \exp(-j2\pi t)}{2}\right)^2$$

$$= 1 + \left(\frac{\exp(j2\pi t) - \exp(-j2\pi t)}{2j}\right) + \left(\frac{\exp(j4\pi t) + \exp(-j4\pi t) + 2}{4}\right)$$

$$= 1.5 + 0.25\exp(-j4\pi t) + 0.5j\exp(-j2\pi t)$$

$$+ 0.25\exp(j4\pi t) - 0.5j\exp(j2\pi t)$$

We thus have

$$c_0 = 1.5$$

$$c_{-1} = -c_1 = -0.5j$$

$$c_{-2} = c_2 = 0.25$$

with all other coefficients equal to zero.

3.17)

When the limits of a definite integral are symmetric with respect to zero, say $T_0/2$ and $-T_0/2$, and the integrand is an odd function, then the value of the integral, i.e., the area under the function, is zero. Also, the product of an odd function and an even function is an odd function. In view of this, when $g(t)$ is a periodic real and even signal and noting a sine function is an odd function, we then have

$$b_n = \left(\frac{2}{T_0}\right) \int_{-T_0/2}^{T_0/2} g(t) \sin\left(\frac{2\pi nt}{T_0}\right) dt = 0$$

when $g(t)$ is a periodic real and odd signal and noting a cosine function is an even function, we then have

$$a_n = \left(\frac{2}{T_0}\right) \int_{-T_0/2}^{T_0/2} g(t) \cos\left(\frac{2\pi nt}{T_0}\right) dt = 0$$

3.18)

In order to determine the quadrature (trigonometric) Fourier series coefficients of $g(t)$, we must use the set of equations (3.34b), (3.34c) and (3.34d). To this effect, the dc value $a_0 = 0$ is zero, $b_n = 0$, as the signal is an even function, and a_n turns out to be as follows:

$$a_n = -\left(\frac{2(1 - \cos(\pi n))}{\pi n}\right)\left(\frac{L}{\pi n}\right)$$

However, we could also use the results obtained in Example 3.16, as the signal in this problem is the integral version of the signal in that example, where the integration constant is zero. We thus have the following:

$$g(t) = \int_{-L}^{L} \left(\sum_{\substack{n=1 \\ n=odd}}^{\infty} \left(\frac{4}{\pi n}\right) \sin\left(\frac{\pi nt}{L}\right) \right) dt = -\sum_{\substack{n=1 \\ n=odd}}^{\infty} \left(\frac{4L}{(\pi n)^2}\right) \cos\left(\frac{\pi nt}{L}\right)$$

3.19)

A signal can be decomposed into real and imaginary parts, we thus have

$$g(t) = \mathrm{Re}[g(t)] + j\mathrm{Im}[g(t)]$$

$$g^*(t) = \mathrm{Re}[g(t)] - j\mathrm{Im}[g(t)]$$

where $*$ represents the complex conjugate operation. From the above equations, the imaginary part can be found, and since $g(t)$ is a real-valued signal, the imaginary part is zero. We thus have:

$$\mathrm{Im}[g(t)] = \left(\frac{1}{2j}\right)\left(g(t) - g^*(t)\right) = 0 \rightarrow g(t) = g^*(t)$$

Using the conjugate function property of the Fourier transform, we have

$$G(f) = G^*(-f) \rightarrow |G(f)| \exp\left(j\measuredangle G(f)\right) = |G(-f)| \exp\left(-j\measuredangle G(-f)\right)$$

From the above, it becomes obvious that the amplitude (magnitude) response $|G(f)|$ is an even function, and the phase response $\measuredangle G(f)$ is an odd function, that is, we have the following:

$$|G(f)| = |G(-f)|$$

$$\measuredangle G(f) = -\measuredangle G(-f)$$

3.20)

Assuming we have

$$g(t) \leftrightarrow G(f)$$

Using time-shifting property, we then have

$$g(t+\beta) \leftrightarrow G(f)\exp(j2\pi\beta f)$$

Using time scaling property, we then have

$$g(\alpha t+\beta) \leftrightarrow \frac{1}{\alpha} G\left(\frac{f}{\alpha}\right) \exp\left(j2\pi\beta\left(\frac{f}{\alpha}\right)\right)$$

Using linearity property, we have

$$\gamma+\mu\, g(\alpha t+\beta) \leftrightarrow \gamma\delta(f)+\frac{\mu}{\alpha} G\left(\frac{f}{\alpha}\right) \exp\left(j2\pi\beta\left(\frac{f}{\alpha}\right)\right)$$

3.21)

A simple proof can be obtained by contradiction. Let's assume that a signal $g(t)$ is simultaneously time-limited to T seconds and band-limited to W Hz. Since $G(f)$ is band-limited, we could view it as the multiplication of $G(f)$ and a signal $P(f)$ whose Fourier transform is rectangular and is also band-limited to W Hz. Note that the inverse Fourier transform of $P(f)$ is $p(t)$ and $p(t)$ is a sinc function which is not time-limited. Using the time-convolution property of the Fourier transform, we know that multiplication in the frequency domain implies convolution in the time domain. The interval for which a convolution is non-zero is the sum of the intervals for which the two convoluted signals are non-zero. Since $p(t)$ is not time-limited, it is impossible to obtain a time-limited signal from the convolution of a time-limited signal with a non-time-limited signal.

3.22)

Using (3.42a), we have

$$G(f)=\int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt=\int_{-\infty}^{\infty} g(t)\cos(2\pi ft)\, dt+\, j\int_{-\infty}^{\infty} g(t)\sin(2\pi ft)\, dt$$

If $g(t)$ is an odd function, then the first integral, which is the real part, becomes zero, as $\cos(2\pi ft)$ is an even function, its multiplication by $g(t)$ is also an odd function, and the limits of the definite integral is symmetric. $G(f)$ is thus pure imaginary.

If $g(t)$ is an even function, then the second integral, which is the imaginary part, becomes zero, as $\sin(2\pi ft)$ is an odd function, its multiplication by $g(t)$ is also an odd function, and the limits of the definite integral is symmetric. $G(f)$ is thus pure real.

3.23)

Multiplication in the time domain implies convolution in the frequency domain. We thus have

$$g_2(t) = g(t)g(t) = G(f) * G(f) = G_2(f)$$

The width property of convolution states that the bandwidth of two convoluted signals is the sum of the bandwidths of the respective signals. The bandwidth of $G_2(f)$ is $2W(= W + W)$ Hz, as the bandwidth of $G(f)$ is W Hz. We apply the convolution property again, and thus have

$$g_3(t) = g(t)g_2(t) = g(t)g(t)g(t) = G(f) * G(f) * G(f) = G_2(f) * G(f)$$

The signal $g_3(t)$ bandwidth $3W(= W + 2W)$. By mathematical induction, the bandwidth of the $g^n(t)$ is thus nW, as we have

$$g^n(t) = g(t)g(t) \dots g(t) = G(f) * G(f) * \dots * G(f)$$

3.24)

Using (3.42a), as well as the linearity and time-shifting properties of the Fourier transform, we can find the Fourier transform of $y(t)$

$$Y(f) = A\,X(f)\exp(-j2\pi fc) + B\,X(f)\exp(-j2\pi fd)$$

We thus have

$$H(f) = \frac{Y(f)}{X(f)} = \frac{A\,X(f)\exp(-j2\pi fc) + B\,X(f)\exp(-j2\pi fd)}{X(f)}$$
$$= A\exp(-j2\pi fc) + B\exp(-j2\pi fd)$$

3.25)

a) Using the definition of the Fourier transform, we have

$$G(f)=\int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt=\int_{-\infty}^{0} e^{at}e^{-j2\pi ft}dt+\int_{0}^{\infty} e^{-at}e^{-j2\pi ft}dt=\frac{1}{a-j2\pi f}+\frac{1}{a+j2\pi f}$$

$$=\frac{2a}{a^2+4\pi^2 f^2}=\frac{\frac{a}{2\pi^2}}{\frac{a^2}{4\pi^2}+f^2}$$

By applying the duality property to the above result, we obtain the following Fourier transform pair:

$$k(t)=\frac{\frac{a}{2\pi^2}}{\frac{a^2}{4\pi^2}+t^2}\leftrightarrow K(f)=e^{-a|f|}$$

Using the scaling property, we get

$$\frac{1}{\frac{a^2}{4\pi^2}+t^2}\leftrightarrow\left(\frac{2\pi^2}{a}\right)e^{-a|f|}$$

Comparing $k(t)$ to $h(t)$, we have $b=\frac{a}{2\pi}$ and thus $a=2\pi b$, using the above Fourier transform, after simplifying it, we get

$$\frac{1}{b^2+t^2}\leftrightarrow\left(\frac{\pi}{b}\right)e^{-2\pi b|f|}$$

b) Let's differentiate $g(t)$ twice with respect to time t, we then get the following:

$$\frac{dg(t)}{dt}=2t(u(t+1)-u(t-1))\rightarrow\frac{d^2g(t)}{dt^2}=2(u(t+1)-u(t-1))$$

Note that the second derivative of $g(t)$ is the well-known rectangular pulse whose width is 2, and we thus have

$$\frac{d^2g(t)}{dt^2} = 2(u(t+1) - u(t-1)) \leftrightarrow 4\ \mathrm{sinc}(2f)$$

Using the differentiation in the time domain property of the Fourier transform, we get

$$g(t) = t^2\big(u(t+1) - u(t-1)\big) \leftrightarrow (j2\pi f)^2 4\mathrm{sinc}(2f) = -(4\pi f)^2\mathrm{sinc}(2f)$$

3.26)

Taking Fourier transform of the relation given, we get

$$4Y(f) + 3\big(j2\pi f Y(f)\big) = X(f) = \frac{1}{1 + j2\pi f}$$

We thus get the Fourier transform of the output signal

$$Y(f) = \frac{1}{1 + j2\pi f} \times \frac{1/3}{4/3 + j2\pi f} = \frac{1}{1 + j2\pi f} - \frac{1}{4/3 + j2\pi f}$$

Taking the inverse Fourier transform, we obtain the output signal $y(t)$

$$y(t) = \left(e^{-t} - e^{-\frac{4t}{3}}\right)u(t)$$

3.27)

We have the following

$$\begin{aligned} y(t) = x(t) + x^2(t) &= m(t)\cos(2\pi f_c t) + (m(t)\cos(2\pi f_c t))^2 \\ &= m(t)\cos(2\pi f_c t) + m^2(t)\left(\frac{1 + \cos(4\pi f_c t)}{2}\right) \\ &= 0.5m^2(t) + m(t)\cos(2\pi f_c t) + 0.5m^2(t)\cos(4\pi f_c t) \end{aligned}$$

The Fourier transform of $y(t)$ is as follows:

$$Y(f) = 0.5M(f) * M(f) + 0.5\big(M(f - f_c) + M(f + f_c)\big)$$
$$+ 0.25(M(f - 2f_c) * M(f - 2f_c) + M(f+2f_c) * M(f+2f_c))$$

Noting that $m(t)$ is band-limited to W and centered around $f = 0$, and $0.5m^2(t)$ is band-limited to $2W$, $Y(f)$ has three major components:

i) The component $M(f) * M(f)$ is a low-pass signal with bandwidth of $2W$ Hz, occupying from $f = 0$ to $f = 2W$.

ii) The component $M(f - f_c) + M(f + f_c)$ is a band-pass signal with a bandwidth of $2W$ Hz, occupying form $f = f_c - W$ to $f = f_c + W$.

iii) The component $M(f - 2f_c) * M(f - 2f_c) + M(f+2f_c) * M(f+2f_c)$ is a band-pass signal with bandwidth of $4W$ Hz, occupying from $f = 2f_c - 2W$ to $f = 2f_c + 2W$.

We can retrieve $m(t)$ if we use a bandpass filter with a mid-frequency of f_c and a bandwidth of $2W$, provided that there are no frequency overlapping, i.e., we have

$$2f_c - 2W > f_c + W \rightarrow f_c > 3W.$$

3.28)

In order for a signal $g(t)$ to have discrete components in its Fourier transform, it must be periodic. To have discrete components at multiples of 5 Hz, the fundamental frequency of the period signal must be 5 Hz or equivalently its period has to be 0.2 seconds. For instance, the following even pulse train satisfies the requirement:

$$g(t) = \begin{cases} 1, & -0.05 \le t \le 0.05 \\ 0, & \text{for the remainder of the period 0.2 seconds} \end{cases}$$

In order for a signal $h(t)$ to have continuous spectrum, it must be non-periodic. To have a bandwidth of 5 Hz, the signal must be not be time-limited. For instance, the following sinc function satisfies the requirement:

$$h(t) = 10\text{sinc}(10t).$$

3.29)

The output signal is the convolution of the two sinc functions, and determining this convolution in the time domain is very difficult in this case. There is an easier and more elegant way to get it, and it is as follows:

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt = 0.25(u(f-2) + u(f+2))$$

and

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}dt = 0.5(u(f-1) + u(f+1))$$

Since we have

$$Y(f) = X(f)H(f) = 0.125(u(f-1) + u(f+1))$$

We can get

$$y(t) = \int_{-\infty}^{\infty} Y(f)e^{j2\pi ft}df = \int_{-1}^{1} 0.125\big(u(f-1) + u(f+1)\big)e^{j2\pi ft}df = 0.25\text{sinc}(2t)$$

3.30)

Using (4.42a), we have

$$G(f) = \int_{-\infty}^{\infty} e^{-\pi t^2}e^{-j2\pi ft}dt$$

Differentiating both sides with respect to the frequency f and using the integration by parts formula gives us the following:

$$\frac{dG(f)}{df} = -j\int_{-\infty}^{\infty}(2\pi t)e^{-\pi t^2}e^{-j2\pi ft}dt = f\int_{-\infty}^{\infty}e^{-\pi t^2}e^{-j2\pi ft}dt = fG(f)$$

We now rearrange the terms and integrate both sides to get $G(f)$:

$$\frac{dG(f)}{G(f)} = fdf \rightarrow \ln G(f) = \frac{1}{2}f^2 + C \rightarrow G(f) = Ke^{-\pi f^2}$$

In order to find the constant K, from the first equation we have $G(0)$ as follows:

$$G(0) = \int_{-\infty}^{\infty}e^{-\pi t^2}dt = 1$$

as the area under the curve is unity, also note that to find the above integral, we used a change of variable. Noting that $G(0) = 1$, we can now find K as follows:

$$G(0) = K = 1$$

We thus have

$$G(f) = e^{-\pi f^2}$$

We conclude therefore that the normalized Gaussian pulse is its own Fourier transform.

3.31)

In an LTI system, the Fourier transform of the output signal $y(t)$ can be obtained as follows:

$$Y(f) = X(f)H(f)$$

where $X(f)$ is the Fourier transform of the input signal $x(t)$ and $H(f)$ is the transfer function, using the frequency translation property of the Fourier transform and noting $1 \leftrightarrow \delta(f)$, we have:

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} x_0 \exp\big(j(2\pi f_0 t + \theta_0)\big)\, e^{-j2\pi ft}dt = x_0 e^{j\theta_0}\delta(f - f_0)$$

We thus have

$$Y(f) = x_0 e^{j\theta_0}\delta(f - f_0)H(f) = x_0 e^{j\theta_0}H(f_0)$$

Noting that we have

$$H(f) = |H(f)|e^{j\varphi(f)}$$

where $|H(f)|$ and $\varphi(f)$ are the magnitude and phase of the transfer function $H(f)$, $Y(f)$ is thus as follows:

$$Y(f) = x_0 e^{j\theta_0}H(f_0) = x_0 e^{j\theta_0}|H(f_0)|e^{j\varphi(f_0)} = x_0|H(f_0)|e^{j(\theta_0+\varphi(f_0))}$$

We can thus conclude that the response of an LTI system to the complex exponential with frequency f_0 is a complex exponential with the same frequency. The amplitude of the response is the product of the input amplitude and $|H(f_0)|$ and its phase is the sum of the input phase and $\varphi(f_0)$.

Complex exponentials are called the eigenfunctions of LTI systems, as they are inputs for which the output is a scaling of the input signal. This also highlights the fact that when the input is a complex exponential, i.e., a sinusoidal signal, the output is also a complex exponential, i.e., a sinusoidal signal with the same frequency.

3.32)

Assuming we have $h(t) = tg(t)$ and $f(t) = \frac{dg(t)}{dt}$, the Cauchy-Schwartz inequality states that

$$\left(\int_{-\infty}^{\infty} t^2 g^2(t)dt\right)\left(\int_{-\infty}^{\infty} \left(\frac{dg(t)}{dt}\right)^2 dt\right) \geq \left(\int_{-\infty}^{\infty} tg(t)\frac{dg(t)}{dt}dt\right)^2 = \left(\frac{1}{4}\right)\left(\int_{-\infty}^{\infty} t\frac{dg^2(t)}{dt}dt\right)^2$$

Integrating the right hand side of the inequality by parts and assuming that $g(t) \to 0$ faster than $\frac{1}{\sqrt{t}} \to 0$, as $t \to \pm\infty$, the right hand side of the equality becomes as follows:

$$\left(\int_{-\infty}^{\infty} t^2 g^2(t)dt\right)\left(\int_{-\infty}^{\infty} \left(\frac{dg(t)}{dt}\right)^2 dt\right) \geq \left(\frac{1}{4}\right)\left(\int_{-\infty}^{\infty} g^2(t)dt\right)^2$$

By re-arranging the above inequality, we have

$$\frac{\int_{-\infty}^{\infty} t^2 g^2(t)dt}{\int_{-\infty}^{\infty} g^2(t)dt} \times \frac{\int_{-\infty}^{\infty} g'^2(t)dt}{\int_{-\infty}^{\infty} g^2(t)dt} \geq \frac{1}{4}$$

Using the properties of the Fourier transform, we have

$$g'(t) = \frac{dg(t)}{dt} \leftrightarrow j2\pi f G(f)$$

$$\int_{-\infty}^{\infty} g^2(t)dt = \int_{-\infty}^{\infty} |G(f)|^2 df$$

The above inequality then becomes as follows:

$$\frac{\int_{-\infty}^{\infty} t^2 g^2(t)dt}{\int_{-\infty}^{\infty} g^2(t)dt} \times \frac{\int_{-\infty}^{\infty} (2\pi f)^2 G^2(f)df}{\int_{-\infty}^{\infty} g^2(t)dt} \geq \frac{1}{4} \to \frac{\int_{-\infty}^{\infty} t^2 g^2(t)dt}{\int_{-\infty}^{\infty} g^2(t)dt} \times \frac{\int_{-\infty}^{\infty} f^2 G^2(f)df}{\int_{-\infty}^{\infty} g^2(t)dt} \geq \frac{1}{16\pi^2}$$

Using (3.67b) and (3.67c), we get

$$(T_{rms})^2(\,B_{rms})^2 \geq \frac{1}{16\pi^2} \rightarrow T_{rms}\,B_{rms} \geq \frac{1}{4\pi}$$

3.33)

For the normalized Gaussian pulse $g(t) = \exp(-\pi t^2)$, we have $G(f) = \exp(-\pi f^2)$. We thus have

$$\int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df = \frac{1}{\sqrt{2}}$$

and using integration by parts, we have

$$\int_{-\infty}^{\infty} t^2 |g(t)|^2 dt = \int_{-\infty}^{\infty} f^2 |G(f)|^2 df = \frac{1}{4\sqrt{2}\pi}$$

Using (3.67a) and (3.67b), we have

$$T_{rms} = \sqrt{\frac{\frac{1}{4\sqrt{2}\pi}}{\frac{1}{\sqrt{2}}}} = \frac{1}{2\sqrt{\pi}}$$

and

$$B_{rms} = \sqrt{\frac{\frac{1}{4\sqrt{2}\pi}}{\frac{1}{\sqrt{2}}}} = \frac{1}{2\sqrt{\pi}}$$

Using (3.67c), we have

$$T_{rms}B_{rms} = \frac{1}{2\sqrt{\pi}} \times \frac{1}{2\sqrt{\pi}} = \frac{1}{4\pi}$$

Not only does the normalized Gaussian signal have the interesting property that both its time and frequency descriptions are of the same functional form, but it has the lowest root-mean square bandwidth and time duration.

3.34)

From Rayleigh's energy theorem, we have

$$\int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df$$

We also know from time-average property of the Fourier transform that

$$\int_{-\infty}^{\infty} g(t) dt = G(0)$$

Using the above results as well as (3.69a) and (3.69b), (3.69c) becomes as follows:

$$T_{NEB}\, B_{NEB} = \frac{\left(\int_{-\infty}^{\infty} |g(t)| dt\right)^2}{\int_{-\infty}^{\infty} |g(t)|^2 dt} \times \frac{\int_{-\infty}^{\infty} |G(f)|^2 df}{2|G(0)|^2} = \frac{\left(\int_{-\infty}^{\infty} |g(t)| dt\right)^2}{2\left|\int_{-\infty}^{\infty} g(t) dt\right|^2}$$

Using Schwartz inequality, we have

$$\int_{-\infty}^{\infty} |g(t)| dt \geq \left|\int_{-\infty}^{\infty} g(t) dt\right| \rightarrow \frac{\int_{-\infty}^{\infty} |g(t)| dt}{\left|\int_{-\infty}^{\infty} g(t) dt\right|} \geq 1$$

The noise-equivalent bandwidth and time duration then becomes as follows:

$$T_{NEB}\, B_{NEB} \geq \frac{1}{2}$$

3.35)

If the signal $g(t)$ is a positive or negative real-valued function of time t for all values of t, then Schwartz inequality as reflected below turns into equality, that is

$$\int_{-\infty}^{\infty}|g(t)|dt \geq \left|\int_{-\infty}^{\infty}g(t)dt\right| \rightarrow \int_{-\infty}^{\infty}|g(t)|dt = \left|\int_{-\infty}^{\infty}g(t)dt\right|$$

As such, we can then have

$$T_{NEB}\,B_{NEB} = \frac{1}{2}$$

3.36)

Assuming a rectangular pulse with a maximum height of $g(0)$ and duration $2T_{AEB}$ whose area is the same as that of $|g(t)|$, we have the following:

$$2T_{AEB}g(0) = \int_{-\infty}^{\infty}|g(t)|dt \geq \int_{-\infty}^{\infty}g(t)dt = G(0) \rightarrow T_{AEB} \geq \frac{G(0)}{2g(0)}$$

Assuming a rectangular pulse with a maximum height of $G(0)$ and duration $2B_{AEB}$ whose area is the same as that of $|G(f)|$, we have the following:

$$2B_{AEB}G(0) = \int_{-\infty}^{\infty}|G(f)|df \geq \int_{-\infty}^{\infty}G(f)df = g(0) \rightarrow B_{AEB} \geq \frac{g(0)}{2G(0)}$$

We thus have

$$T_{AEB}B_{AEB} \geq \frac{1}{4}$$

3.37)

If the signal $g(t)$ is a positive or negative real-valued function of time t for all values of t, and also its Fourier transform $G(f)$ is a positive or negative real-valued function of frequency f for all values of f, we then have

$$\int_{-\infty}^{\infty} |g(t)| dt = \int_{-\infty}^{\infty} g(t) dt$$

and

$$\int_{-\infty}^{\infty} |G(f)| df = \int_{-\infty}^{\infty} G(f) df$$

As such, we can then have

$$T_{AEB}\, B_{AEB} = \frac{1}{4}$$

3.38)

The transfer function of this filter is real, as such it does not introduce any delay. This filter is a low-pass filter as its magnitude response is a decreasing function of the frequency f, and its maximum value is 1, when $f = 0$. The 3-dB bandwidth is the frequency for which the magnitude response is $\frac{1}{\sqrt{2}}$ times the maximum of the magnitude response. We thus have

$$\left(1 + \left(\frac{f}{10000}\right)^2\right)^{-0.5} = \frac{1}{\sqrt{2}} \times 1 \rightarrow f = 10{,}000$$

The Fourier transform of the sinusoidal input signal $x(t)$, whose frequency is 20,000 Hz, is as follows:

$$X(f) = 3.5\left(e^{\frac{j\pi}{4}}\delta(f - 20000) + e^{-\frac{j\pi}{4}}\delta(f + 20000)\right)$$

We thus have the Fourier transform of the output signal as follows:

$$Y(f) = X(f)H(f) = 3.5\left(e^{\frac{j\pi}{4}}\delta(f-20000) + e^{-\frac{j\pi}{4}}\delta(f+20000)\right)\left(1+\left(\frac{f}{10000}\right)^2\right)^{-0.5}$$

$$= 3.5\left(e^{\frac{j\pi}{4}}\delta(f-20000) + e^{-\frac{j\pi}{4}}\delta(f+20000)\right)\left(1+\left(\frac{20000}{10000}\right)^2\right)^{-0.5}$$

$$= \left(\frac{3.5}{\sqrt{5}}\right)\left(e^{\frac{j\pi}{4}}\delta(f-20000) + e^{-\frac{j\pi}{4}}\delta(f+20000)\right)$$

By taking the inverse Fourier transform, we have:

$$y(t) = \frac{7}{\sqrt{5}}\cos\left(40000\pi t + \frac{\pi}{4}\right).$$

Only the amplitude of the input signal has changed. This is an example that shows in an LTI system, we always have sinusoid in sinusoid out.

3.39)

Due to the linearity property and the convolution in the time domain property of the Fourier transform, we have

$$Y(f) = X(f) + X(f) * X(f)$$

Due to the duality property, we also know that

$$X(f) = \int_{-\infty}^{\infty} 2\text{sinc}(2t)e^{-j2\pi f t}dt = U(f+1) - U(f-1)$$

where $U(f)$ is a step function. We can now find the convolution of $X(f)$ with itself:

$$X(f) * X(f) = (-|f| + 2)(U(f+2) - U(f-2))$$

Therefore, we have

$$Y(f) = \big(U(f+1) - U(f-1)\big) + \Big(\big(-|f| + 2\big)\big(U(f+2) - U(f-2)\big)\Big)$$

We have both out-of-band frequencies (outside the frequency range of the input signal), which can be easily filtered out and in-band frequencies (within the frequency range of the input signal) which cannot be filtered out, thus yielding distortion.

3.40)

Noting that for a Gaussian signal, we have

$$e^{-\pi t^2} \leftrightarrow e^{-\pi f^2}$$

and using the scaling property of the Fourier transform, we have

$$\exp\left(-\frac{t^2}{2\sigma^2}\right) \leftrightarrow 2\sigma^2 \exp(-2\sigma^2 f^2)$$

The power spectral density of the nonperiodic power signal is as follows:

$$S(f) = 2\sigma^2 \exp(-2\sigma^2 f^2)$$

The average power content is as follows:

$$R(0) = 1$$

3.41)

Using (3.97), the autocorrelation function is as follows:

$$R_g(\tau) = 0.5\cos(10\pi\tau) + 0.5\cos(20\pi\tau)$$

Using (3.98), we can get its power spectral density:

$$S_g(f) = 0.25\big(\delta(f-5) + \delta(f+5) + \delta(f-10) + \delta(f+10)\big)$$

3.42)

Lack of full carrier synchronization in a QAM system, i.e., an error in the phase and/or the frequency of the carrier in a QAM modulator, results in co-channel interference. In other words, the carrier in the demodulator has a phase offset θ.

The input to the upper low-pass filter is thus as follows:

$$\begin{aligned} 2s(t)\cos(2\pi f_c t + \theta) &= 2(m_1(t)\cos(2\pi f_c t) + m_2(t)\sin(2\pi f_c t))\cos(2\pi f_c t + \theta) \\ &= m_1(t)\cos(\theta) - m_2(t)\sin(\theta) + m_1(t)\cos(4\pi f_c t + \theta) \\ &+ m_2(t)\sin(4\pi f_c t + \theta) \end{aligned}$$

The output at the upper low-pass filter is then as follows:

$$m_1(t)\cos(\theta) - m_2(t)\sin(\theta)$$

Since $\theta \neq 0$, the signal in the in-phase channel has the scaled down version of both the desired signal $m_1(t)$ and the unwanted signal $m_2(t)$. This is known as co-channel interference. A similar effect also occurs in the quadrature channel.

3.43)

Using Figure 3.41, we have

$$s(t) = m_1(t)\cos(2\pi f_c t) + m_2(t)\sin(2\pi f_c t)$$

Noting multiplication of a function by a sine wave or a cosine wave in the time-domain implies shifts in the frequency domain, we have